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On a rotational flow disturbed by gravity

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We examine a rapidly rotating flow that exhibits periodic vortex detachment. Specifically, the rotation/symmetry axis of a fluid-filled cylinder is set perpendicular to gravity. A free buoyant cylindrical float placed within the container is acted upon by both centrifugal and gravitational forces, the competition of which causes fluid motion and, in certain parameter ranges, flow instability. The motion is determined, a criterion for separation is advanced, preliminary experiments and data are described and the relationship of this phenomenon to other examples of vortex shedding in rotating fluids is discussed.

1. Introduction

Phillips (1960) studied the motion of a fluid that incompletely fills a cylindrical container in rapid rotation about its symmetry axis, which is perpendicular to the direction of gravity. A central air core develops as a consequence of centrifugal forces overcoming buoyancy forces but gravity can cause an instability which is manifested as a column collapse. Phillips reasoned that a necessary condition for the stability of the flow is that the radial pressure gradient be positive everywhere at the surface of the bubble and on this basis he derived the stability criterion

$$g/\Omega^2 r < \tfrac{1}{3}, \tag{1}$$

where r is the radius of the air column, g is the gravitational acceleration and Ω is the rotation rate of the container.

Experiments with a partially filled container yielded results that were consistent with this formula and the flow became unstable when the radius of the core was decreased to a 'critical' value. The instability was described by Phillips simply as column collapse, and indeed the phenomenon appeared to be a very complicated interaction of waves, viscous and nonlinear effects.

Gans (1976) examined more closely the nature of this instability within an almost completely filled cylinder as an off-shoot of an investigation (with W. V. R. Malkus) of unstable precessional motions in a rotating cylinder. In both of these experimental configurations a periodic formation of very intense line vortices was observed near the centre, whose axes were parallel to the main rotation axis.

In another quite different context, Fujita (1971) reported the periodic production of strong vortices about the main funnel of a tornado that is in translation along the ground.

These phenomena are so strikingly similar that something more than mere coincidence is suggested. It would appear that in each case the vortices are shed from the region of the central core as a manifestation of shear-layer separation in much the same way as two-dimensional vortices are detached from a *solid* cylinder that rotates at constant speed in a uniformly moving stream (see Goldstein 1938). Further support for this view comes from the work of Walker & Stewartson (1974), who examined separation in the free shear layer of the Taylor column formed by a body moving slowly in a rotating fluid.

As an illustration of the periodic generation of vortices in rotating flows and in order to begin the study of this process in a definite setting which is as simple as possible, we consider the problem treated by Phillips with one important modification: the air column is replaced by a very long, very buoyant, rigid cylinder, an ordinary plastic drinking straw which is sealed at both ends. This change eliminates centrifugal waves on the interface, column deformation, and collapse, and yet it retains the phenomena of instability and vortex shedding. In addition, measurements are much simplified as is the mathematical analysis (although this is never really apparent).

An exploratory experiment immediately reveals the following information:

(*a*) At high rotation rates, the fluid motion is stable and steady. The straw 'floats' near centre, displaced downward a bit, and it rotates at a constant rate Ω_s that is slightly less than the angular velocity Ω of the container, i.e. $\Omega_s < \Omega$. Relative to the tank, the straw has a retrograde angular velocity.

(*b*) The difference $\Omega - \Omega_s$ increases significantly as Ω decreases.

(*c*) At a critical value of Ω which depends on the diameter of the straw, the motion becomes unstable, unsteady and quasi-periodic. In cyclic fashion, an intense line vortex, figure 1 (plate 1), is shed from the vicinity of the straw which thereupon experiences an immediate and rather large increase in its angular velocity Ω_s. The free vortex decays in the time required for spin-up; meanwhile, the rotation rate of the straw again decreases markedly until another vortex is shed and the process is repeated.

It is the object of this research to determine the motion of the fluid, the position and angular velocity of the straw and to advance a tentative criterion for the onset of instability. The broader issue, that rapidly rotating flows through their strong propensity for two-dimensionality and longitudinal rigidity can indeed experience periodic vortex separation as do solid bodies, will be pursued independently.

2. Formulation

Certain idealizations are introduced at the outset in order to facilitate a very extensive calculation. We consider then the rapid rotation of an infinitely long, fluid-filled cylinder about its symmetry axis, which is aligned perpendicular to gravity as shown in figure 2. An infinitely long, weightless, cylindrical float, i.e. a sealed straw of radius r_1, is placed inside the container, where it is free to move in response to gravitational, pressure and viscous forces. The position of the weightless straw is such that the net force and torque on it are zero at all times; however, we shall be concerned here only with steady-state equilibrium.

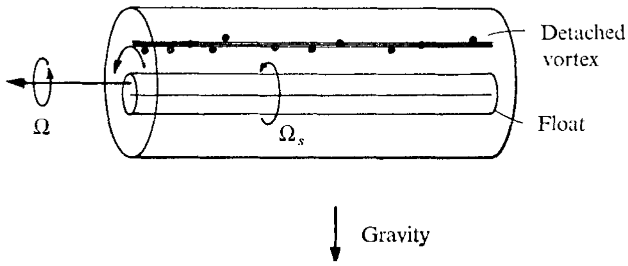

FIGURE 2. Experimental configuration, showing the orientation of axes, the position of the float and a detached vortex.

In an inertial frame the equations of steady motion of a viscous incompressible fluid of density ρ_w and kinematic viscosity ν are

$$\nabla \cdot \mathbf{q} = 0,$$

$$(\nabla \times \mathbf{q}) \times \mathbf{q} = -\nabla (p/\rho_w + gy + \tfrac{1}{2}\mathbf{q} \cdot \mathbf{q}) - \nu \nabla \times \nabla \times \mathbf{q}. \tag{2}$$

The boundary condition on the outer cylindrical surface $r = r_2$ is

$$\mathbf{q} = \Omega r_2 \hat{\boldsymbol{\theta}}$$

whereas on the surface of the straw, given by

$$(x - \epsilon r_1 \cos \omega)^2 + (y - \epsilon r_1 \sin \omega)^2 = r_1^2,$$

the velocity is

$$\mathbf{q} = \Omega_s r_1 \hat{\mathbf{k}} \times \hat{\mathbf{n}}.$$

(Here $\hat{\mathbf{n}}$ is the unit normal vector at the inner cylinder.) The parameters ϵ, ω and Ω_s, which specify the position and angular velocity of the straw, must be determined from the motion and the additional conditions that the net force and torque on this body are zero. These requirements are given in mathematical form in (17) and (19) below, after the selection of a convenient system of co-ordinates.

A stream function ψ such that

$$\mathbf{q} = \nabla \times [\psi(x,y)\, \hat{\mathbf{k}}] \tag{3}$$

can be introduced since, in the idealized geometry, the motion is two-dimensional and independent of distance along the rotation axis.

If distance, velocity and pressure are scaled by r_1, Ωr_1 and $\rho_w \Omega^2 r_1^2$, the problem can be cast in dimensionless terms. Equation (2) is then

$$-(\Delta\psi)(\nabla\psi) = -\nabla(p + \alpha y + \tfrac{1}{2}\nabla\psi \cdot \nabla\psi) + E\nabla(\Delta\psi) \times \hat{\mathbf{k}}, \tag{4}$$

where

$$\alpha = g/\Omega^2 r_1, \quad E = \nu/\Omega r_1^2. \tag{5}$$

Of these, α is the inverse Froude number and E is the usual Ekman number, both of which are small in fluid motions examined herein.

Although (4) must be used to compute the pressure force on the straw, it is generally simpler to consider the equation for vorticity as primary:

$$[\nabla(\Delta\psi)] \times \nabla\psi = E\hat{\mathbf{k}}\Delta\Delta\psi. \tag{6}$$

The problem now is to solve these difficult equations in a two-dimensional domain bounded by non-concentric circles; see figure 3(*a*).

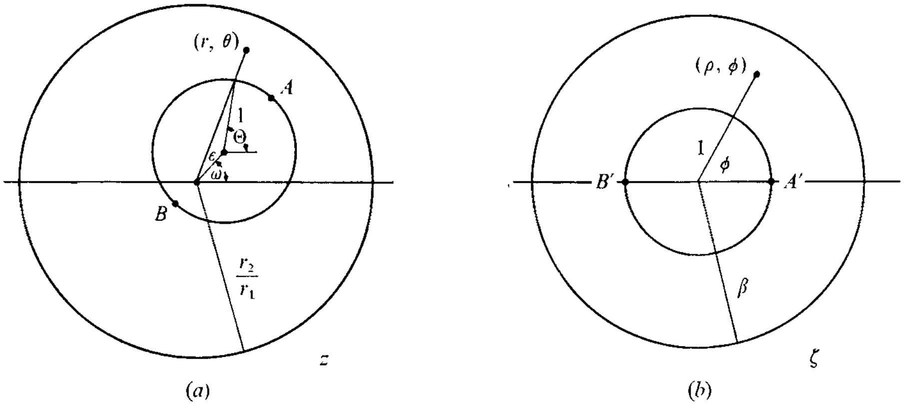

FIGURE 3. Cross-section of the container and the mapping onto concentric cylinders in the ζ plane.

This task is eased somewhat by a change of variables that transforms the geometry into a circular annulus, i.e. the cylinders are made concentric. Although the equations of motion are complicated by this manoeuvre, the trade-off is advantageous as shown by Wood (1957) and Segel (1961), who both solved similar problems of viscous flow between *fixed* non-concentric cylinders. The analysis presented here is an adaptation and an extension of that employed by Wood, who also dealt with rapid rotation (small Ekman number). With

$$z = x + iy = r e^{i\theta}$$

and

$$\zeta = \rho e^{i\phi}$$

the desired transformation is

$$\zeta = \frac{z + (\delta - \epsilon) e^{i\omega}}{\delta z + (1 - \delta\epsilon) e^{i\omega}} \tag{7}$$

where

$$\delta = -2\epsilon \left[\left(\frac{r_2}{r_1}\right)^2 - 1 - \epsilon^2 + \left\{ \left(\left(\frac{r_2}{r_1}\right)^2 - 1 - \epsilon^2 \right)^2 - 4\epsilon^2 \right\}^{\frac{1}{2}} \right]^{-1}. \tag{8}$$

The interior circle in the x, y plane (the straw) becomes $\rho = 1$ in the ζ plane (see figure 3) and the outer boundary $|z| = r_2/r_1$ maps into the concentric circle

$$\rho = \beta = (r_2/r_1)(1 - \epsilon\delta)^{-1}. \tag{9}$$

In terms of the Jacobian of this transformation

$$J = J(\rho, \phi) = \frac{(1-\delta^2)^2}{(1 - 2\delta\rho\cos\phi + \delta^2\rho^2)^2} \tag{10}$$

the radial and azimuthal velocity components in the ζ plane are

$$u = \frac{1}{\rho J^{\frac{1}{2}}} \frac{\partial\psi}{\partial\phi}, \quad v = -\frac{1}{J^{\frac{1}{2}}} \frac{\partial\psi}{\partial\rho}. \tag{11}$$

The equations of motion (4) and (5) become

$$-J^{-1}\Delta\psi\nabla\psi = -\nabla(P + \alpha y + \tfrac{1}{2}J^{-1}\nabla\psi . \nabla\psi) + \mathrm{E}\nabla(J^{-1}\Delta\psi) \times \hat{\mathbf{k}}, \tag{12}$$

$$\frac{1}{\rho} \frac{\partial(J^{-1}\Delta\psi, \psi)}{\partial(\rho, \phi)} = E\Delta(J^{-1}\Delta\psi). \tag{13}$$

The boundary conditions at the impermeable walls are then

$$\psi = 0, \quad \frac{\partial \psi}{\partial \rho} = -\frac{\Omega_s}{\Omega}[J(1,\phi)]^{\frac{1}{2}} \quad \text{on} \quad \rho = 1; \tag{14}$$

$$\psi = Q \text{ (a constant)}, \frac{\partial \psi}{\partial \rho} = -\frac{r_2}{r_1}[J(\beta,\phi)]^{\frac{1}{2}} \quad \text{on} \quad \rho = \beta. \tag{15}$$

The stress components in the new co-ordinates are

$$\left.\begin{aligned} P_{\rho\rho} &= -p + \frac{2E}{J^{\frac{1}{2}}}\left(\frac{\partial u}{\partial \rho} + \frac{v}{\rho}\frac{\partial}{\partial \phi}J^{\frac{1}{2}}\right), \\ P_{\phi\rho} &= E\left(\rho\frac{\partial}{\partial \rho}\left(\frac{v}{\rho J^{\frac{1}{2}}}\right) + \frac{1}{\rho}\frac{\partial}{\partial \phi}\left(\frac{u}{J^{\frac{1}{2}}}\right)\right). \end{aligned}\right\} \tag{16}$$

(Here P_{ab} is the component in the direction of b increasing, of the stress exerted at (a, b) across the surface a = constant.) The condition of no net force on the weightless straw $\rho \leqslant 1$ may be expressed in terms of vector components (see Segel 1961):

$$x \text{ component:} \quad \int_0^{2\pi} (P_{\rho\rho}\cos\Theta - P_{\rho\phi}\sin\Theta)\, J^{\frac{1}{2}} d\phi = 0;$$

$$y \text{ component:} \quad \int_0^{2\pi} (P_{\rho\rho}\sin\Theta + P_{\rho\phi}\cos\Theta)\, J^{\frac{1}{2}} d\phi = 0;$$

where $z - \epsilon e^{i\omega} = e^{i\Theta}$, as illustrated in figure 3. These equations can be combined by multiplying the second by i and adding it to the first to obtain

$$\int_0^{2\pi} (P_{\rho\rho} + iP_{\rho\phi})\, e^{i\Theta} J^{\frac{1}{2}} d\phi = 0. \tag{17}$$

According to (7), the function $e^{i\Theta}$ is given by

$$e^{i\phi} = \frac{e^{i(\Theta-\omega)} + \delta}{\delta e^{i(\Theta-\omega)} + 1}$$

and since $e^{-i\omega}$ is a constant, (17) can be written as

$$\int_0^{2\pi} (P_{\rho\rho} + iP_{\rho\phi})\frac{e^{i\phi} - \delta}{1 - \delta e^{i\phi}} J^{\frac{1}{2}} d\phi = 0. \tag{18}$$

Finally there must be no net torque on the straw, the mathematical statement being

$$\int_0^{2\pi} P_{\rho\phi} J^{\frac{1}{2}} d\phi = 0. \tag{19}$$

The formulation is now complete and the solution of (12) and (13) must be determined subject to the boundary conditions (14) and (15) and the constraints (18) and (19). All unknowns – the fluid velocity $\mathbf{q}$, the pressure p and the location and rotation rate of the straw as specified by ϵ, ω and Ω_s – must be expressed in terms of position (x, y) and/or the basic parameters of motion α, E and r_2/r_1.

3. Procedure

A description of the general approach and a summary of results are presented in this section; important details of analysis appear in appendices or in cited references.

A satisfactory solution of the problem formulated in the last section, especially the calculation of the angular velocity of the straw Ω_s, involves both viscous and nonlinear processes. Fortunately the inherent difficulties of the full Navier–Stokes equations are partially mitigated in this problem of rapid rotation because parameters α and E are both small numbers. We can then attempt to represent all unknown quantities as regular perturbation expansions (essentially in powers of α) about some known state and further to use boundary-layer (singular perturbation) theory to analyse regions of significant viscous shear.

We are concerned primarily with a configuration in which r_2/r_1 is large so that the straw actually occupies very little of the total volume of the cylindrical container. In this case (8) and (9) imply

$$\beta \simeq r_2/r_1, \quad \delta \simeq -\epsilon/\beta^2;$$

i.e. β is large and δ is very small (and negative). As Wood (1957) noted it is then advantageous to use δ as the basic expansion parameter and to regard β and E as independent parameters. Therefore, let

$$\left.\begin{array}{lrl} \text{(i)} & \psi &= \psi_0+\delta\psi_1+\delta^2\psi_2+\ldots; \\ \text{(ii)} & p &= p_0+\delta p_1+\delta^2 p_2+\ldots; \\ \text{(iii)} & \Omega_s/\Omega &= \gamma_0+\delta\gamma_1+\delta^2\gamma_2+\ldots; \\ \text{(iv)} & \omega &= \omega_0+\delta\omega_1+\delta^2\omega_2+\ldots; \\ \text{(v)} & \alpha &= \qquad \delta\alpha_1+\delta^2\alpha_2+\ldots; \\ \text{(vi)} & Q &= Q_0+\delta Q_1+\delta^2 Q_2+\ldots . \end{array}\right\} \tag{20}$$

In this format

$$\left.\begin{array}{lrll} \text{(i)} & J &= J_0+\delta J_1 \qquad +\delta^2 J_2+\ldots, & \\ & &= 1+\delta(4\rho\cos\phi)+\delta^2(12\rho^2\cos^2\phi-2\rho^2-2) & +\ldots; \\ \text{(ii)} & \epsilon &= \quad -\delta(\beta^2-1) & +\ldots; \\ \text{(iii)} & r_2/r_1 &= \beta \qquad\qquad +\delta^2\beta(\beta^2-1) & +\ldots . \end{array}\right\} \tag{21}$$

These expansions are now substituted into the basic equations (12) and (13) as well as the boundary conditions (14) and (15) and constraints (18) and (19). This is a fairly straightforward but very laborious task, the result of which is a sequence of boundary-value problems for the perturbation quantities associated with the same power of δ. The object is to carry the analysis forward until a sufficiently good approximation for each dependent variable in (20) is attained. Ordinarily we would be content to find all terms of first order. Unfortunately there is no retrograde rotation in evidence at this stage of theoretical development and it is necessary to proceed to the second-order calculation of the torque. Since the shear stress exerted on the straw by the surrounding fluid is sensitively related to the form and variation of flow in the boundary layer, the computation of torque on the inner cylinder requires a fairly precise determination of rectified zonal currents, i.e. the azimuthally independent component of flow within the boundary layer. As might be suspected the algebra involved is extensive and tedious. For this reason, the analysis at second order is aimed directly at the computation of γ_2 and the circumferential velocity at $\rho = 1+$, the outer edge of the boundary layer. This information allows us to compute the angular velocity of the straw and to advance a plausible criterion for instability.

The outcome of this calculation is surprising and important because the desired corrections are found to be much larger than expected, $O(\alpha^2/E^{\frac{1}{2}})$ rather than $O(\alpha^2)$. Since E is small, terms of this magnitude are, practically speaking, comparable to those of first order. The formal procedure is however based on the restriction $\alpha \ll E^{\frac{1}{2}}$.

The main results of the analysis detailed in appendix B that concern the position and motion of the cylindrical float are the formulae

$$\left.\begin{array}{lrl} \text{(i)} & \Omega_s/\Omega = & 1-\sqrt{2}\,\delta^2 E^{-\frac{1}{2}}\beta^2(\beta^2-4)+\ldots; \\ \text{(ii)} & \omega = & -\frac{1}{2}\pi+\ldots; \\ \text{(iii)} & \epsilon = & -\delta(\beta^2-1)+\ldots; \\ \text{(iv)} & \alpha = & -2\delta\beta^2+\ldots; \end{array}\right\} \tag{22}$$

$$\partial\psi(1+,\phi)/\partial\rho = -1+2\delta(\beta^2-1)\cos\phi+\sqrt{2}\,E^{-\frac{1}{2}}\delta^2\beta^2(\beta^2-4)+\ldots. \tag{23}$$

The last expression is essentially the inviscid, circumferential component of velocity v, just outside the boundary layer on the straw.

4. Discussion

Since $\beta\ (\simeq r_2/r_1)$ is a large number, (22) and (23) can easily be rewritten in terms of the basic parameters

$$\alpha = g/(\Omega^2 r_1), \quad E = \nu/(\Omega r_1^2),$$

and it follows that

$$\left.\begin{array}{lrl} \text{(i)} & \Omega_s/\Omega \simeq & 1-2^{-\frac{3}{2}}\alpha^2E^{-\frac{1}{2}}; \\ \text{(ii)} & \omega \simeq & -\frac{1}{2}\pi; \\ \text{(iii)} & \epsilon \simeq & \frac{1}{2}\alpha; \end{array}\right\} \tag{24}$$

and

$$v(1+,\phi) \simeq 1+\alpha\cos\phi-2^{-\frac{3}{2}}\alpha^2E^{-\frac{1}{2}}. \tag{25}$$

Equations (24) describe the position and angular velocity of the straw. According to (i) of this group, the rotation rate of the straw is less than that of the cylinder and, relative to the tank, the float has a retrograde velocity. From (ii) and (iii) we learn that the centre of the straw is displaced downward a dimensional distance $\frac{1}{2}\alpha r_1$.

According to Phillips (1963), fluid particles convert potential energy to kinetic energy in moving, that is falling, from the top side to the underside of the float. Since the circumferential velocity is increased below the free core, continuity requires that the fluid thickness there decrease, so that the displacement is downward as predicted. In this discussion, we shall need no more precise description of the fluid motion about the non-concentric cylinders other than it is a complicated perturbation from rigid rotation with a boundary layer at each wall.

The formula for Ω_s is the easiest to try to confirm since a constant rotation rate can be determined precisely with a stroboscope. However, measurement of position requires more sophisticated techniques and was not attempted beyond visual corroboration that the displacement was indeed a small distance downward.

Preliminary experiments were carried out in a Plexiglas cylinder of inner radius

7·6 cm and length 19 cm. Ordinary drinking straws sealed at both ends were used as floats. Individual diameters varied from 0·18 to 0·4 cm but the lengths were fixed at approximately 18 cm. For safety reasons, the rotation rate of the apparatus was kept below 2400 r.p.m. although higher angular velocities were attainable by the variable speed motor. In a typical run, Ω was set at the upper limit and measurement of the steady-state value of Ω_s was made after ten to fifteen minutes had elapsed. The rotation rate Ω was then decreased by 100 r.p.m. increments and the procedure repeated until instability occurred. The point of instability thus bounded within the last 100 r.p.m. interval of measurement was more accurately fixed by traversing this range several times from above and below in increments of 20 r.p.m.

The results shown in figure 4 are consistent and in fair agreement with theory, equation (24i), but also indicate the need for better data. Some of the sources of error are as follows:

(*a*) The tank and the straws are too short to be considered infinitely long. Viscous end effects are probably significant since these corrections increase the observed value of Ω_s.

(*b*) The restricted range of rotation rates is one for which α is not sufficiently small to satisfy the assumptions made in the analysis. Typical values are $0{\cdot}05 \leqslant \alpha \leqslant 0{\cdot}4$, and $0{\cdot}01 \lesssim E^{\frac{1}{2}} \lesssim 0{\cdot}05$.

(*c*) The sealed drinking straws were not weightless (a density of 0·2 g/cm^3 was typical), exactly circular or rigid.

(*d*) At certain frequencies, resonant inertial modes may have been excited.

Boundary-layer separation and the detachment of a vortex from the surface of a body in a rotational flow are very complicated phenomena that are not readily susceptible to theoretical study. An adverse tangential pressure gradient and a point of zero shear stress on the surface of a body are the most commonly adopted criteria for separation; Phillips' condition for the stability of a central air column in rotating fluid, equation (1), requires the net radial pressure gradient to be positive everywhere on the bubble surface.

We take the onset of instability to coincide with the appearance of the first stagnation point in the inviscid flow at the outer edge of the boundary layer on the surface of the straw. This point of zero absolute velocity is closely related to flow reversal within the neighbouring boundary layer and the 'suction' of that layer off the surface of the float. Since the effects of both adverse tangential and normal pressure gradients are taken into account, this criterion should provide a reasonable quantitative (but conservative) estimate for the onset of instability. This calculation is moderately involved even though the velocity is approximated by only three terms of its complete series expansion. Since the main features of the flow are incorporated at this stage, and the values of α at instability are small, the approximation seems valid and adequate.

The normal component of velocity u is, to lowest order, zero at the straw and throughout the shear layer surrounding this surface. Therefore, stagnation points are situated at $v(1+,\phi) = 0$ or by (25)

$$v(1+,\phi) = 1+\alpha\cos\phi - 2^{-\frac{3}{2}}\alpha^2 E^{-\frac{1}{2}} = 0.$$

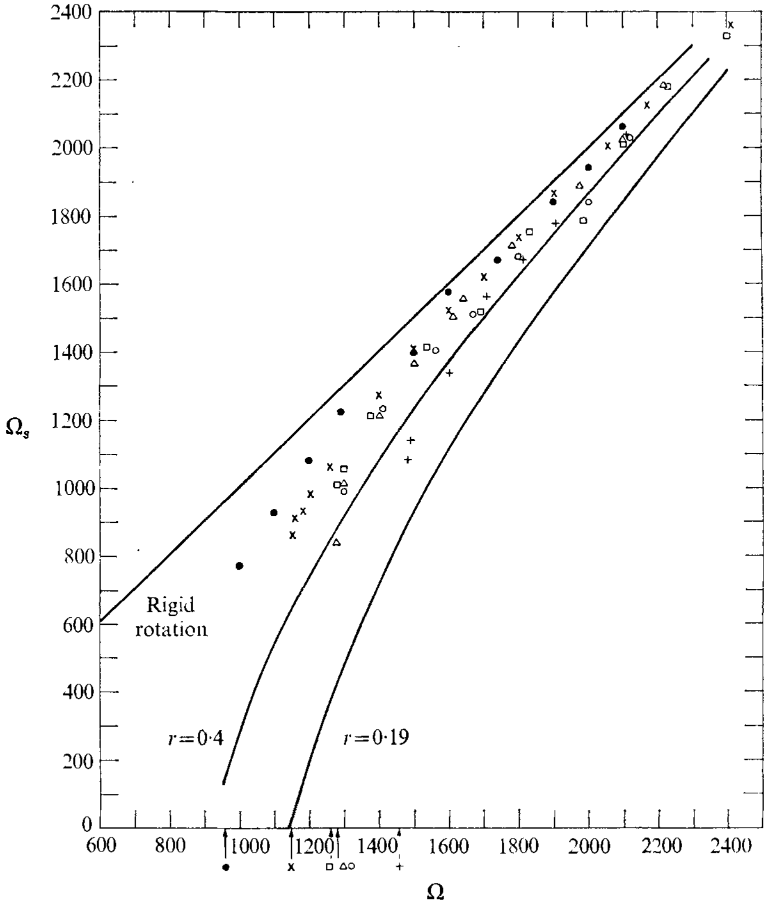

FIGURE 4. Rotation rate of the straw Ω_s *vs.* the angular velocity of the container Ω in several cases of varying diameter: ●, $r_1 = 0{\cdot}4$ cm, $L = 16$ cm, $\rho_s = 0{\cdot}44$ g/cm^3; ×, $r_1 = 0{\cdot}33$ cm, $L = 18$ cm, $\rho_s = 0{\cdot}15$ g/cm^3; △, $r_1 = 0{\cdot}29$ cm, $L = 17{\cdot}6$ cm, $\rho_s = 0{\cdot}17$ g/cm^3; □, $r_1 = 0{\cdot}2$ cm, $L = 18{\cdot}1$ cm, $\rho_s = 0{\cdot}18$ g/cm^3; ○, $r_1 = 0{\cdot}18$ cm, $L = 18$ cm, $\rho_s = 0{\cdot}27$ g/cm^3; +, $r_1 = 0{\cdot}13$ cm, $L = 17{\cdot}7$ cm, $\rho_s = 0{\cdot}12$ g/cm^3. Theoretical curves shown correspond to $r_1 = 0{\cdot}4$ cm and $r_1 = 0{\cdot}19$ cm. The value of Ω at which instability occurred in each case is shown by an arrow on the horizontal axis.

For Ω large but decreasing, and r_1 fixed, the point of stagnation occurs first at $\phi = \pi$ ($\theta = \frac{1}{2}\pi$ in the physical plane) when the rotation rate reaches a critical value given by

$$1 - \alpha - 2^{-\frac{3}{2}}\alpha^2 E^{-\frac{1}{2}} = 0. \tag{26}$$

This formula defines the transition border between stable and unstable regimes. (A description of the unsteady unstable flow is given in § 1.) The stability boundary line can be plotted as α *vs.* $E^{\frac{1}{2}}$ or r_1 *vs.* Ω, the latter being more relevant to the acquisition and interpretation of data. If r_1 is prescribed, the critical value of Ω below which the flow is unstable can be read directly from figure 5. Conversely, if Ω is specified the graph tells how large the radius of the straw must be in order to have a stable state of motion. The few experimental points indicate fair agreement with (26) (or (1) for that matter) but do not cover a sufficiently broad

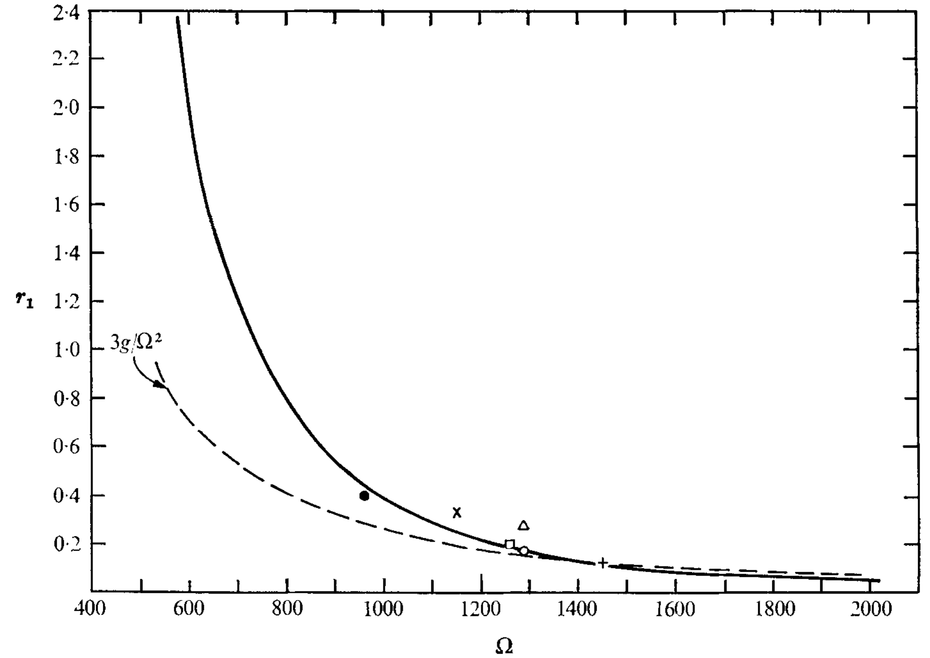

FIGURE 5. Radius of the straw (cm) *vs.* angular velocity at which instability occurs as determined from theory, equation (26) (solid curve). Experimental results are indicated, as is the criterion of Phillips for an unstable air core (dashed curve).

range of parameter values to be conclusive evidence. More precise measurements will be required to evaluate this stability condition.

As to the speculations made in the introduction, they remain just that for the present. The rigidity and two-dimensionality of rapidly rotating flows do seem to cause periodic vortex 'detachment' very like that about a solid body, but undoubtedly it will be difficult to obtain a quantitative statement of the general conditions for, and periodicity of, vortex shedding other than 'shear-layer instability'. A slight variant of the mechanism described here is evidently involved in the air core collapse as well (Gans 1976), and may also contribute to the production of vortices in many other rotating fluid motions.

The study of Walker & Stewartson (1974) is important in this regard because it shows that separation can occur at a fluid interface in a rotating flow. It should be noted that although the production of longitudinal vortices is known to be a part of the process by which such Taylor columns are detached when the fluid flow is slightly nonlinear (Taylor 1923), the periodic shedding of vortices from a column that acts like a solid cylinder has not been observed. Quite possibly the causative mechanism in this situation is too sensitive to do much towards re-establishing the basic flow. However, Taylor columns formed in other ways, say by a source–sink arrangement, might be more 'rigid', in which case rhythmic detachment of the quasi-stable state could be detected and analysed.

Flow past a vortex, or a vortex in motion, could well involve a hierarchy of detached vortices. Strong vortices shed from the primary swirl would in turn detach their own satellite vortices by the same physical process and this would continue until some criterion for separation relating local translational and rotational velocities is violated. This mechanism may be involved in the production of suction spots reported by Fujita (1971), and any possible implication in the formation of a tornado from a larger, cyclonic cloud should be examined.

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Appendix A

From the definitions (11), (16) and (20) it follows that

$$u = \frac{1}{\rho J^{\frac{1}{2}}}\frac{\partial\psi}{\partial\phi} = u_0+\delta u_1+\delta^2 u_2+\dots$$

$$= \frac{1}{\rho}\left[\frac{\partial\psi_0}{\partial\phi}+\delta\left(\frac{\partial\psi_1}{\partial\phi}-2\rho\cos\phi\frac{\partial\psi_0}{\partial\phi}\right)+\delta^2\left((\rho^2+1)\frac{\partial\psi_0}{\partial\phi}-2\rho\cos\phi\frac{\partial\psi_1}{\partial\phi}+\frac{\partial\psi_2}{\partial\phi}\right)\right]+\dots;$$

$$v = -\frac{1}{J^{\frac{1}{2}}}\frac{\partial\psi}{\partial\rho} = v_0+\delta v_1+\delta^2 v_2+\dots$$

$$= -\left[\frac{\partial\psi_0}{\partial\rho}+\delta\left(\frac{\partial\psi_1}{\partial\rho}-2\rho\cos\phi\frac{\partial\psi_0}{\partial\rho}\right)+\delta^2\left(\frac{\partial\psi_2}{\partial\rho}-2\rho\cos\phi\frac{\partial\psi_1}{\partial\rho}+(\rho^2+1)\frac{\partial\psi_0}{\partial\rho}\right)\right]+\dots;$$

$$P_{\rho\rho} = P^{(0)}_{\rho\rho}+\delta P^{(1)}_{\rho\rho}+\delta^2 P^{(2)}_{\rho\rho}+\dots,$$

where

$$P^{(0)}_{\rho\rho} = -p_0+2E\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\psi_0}{\partial\phi}\right),$$

$$P^{(1)}_{\rho\rho} = \left[-p_1+2E\left\{\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\psi_1}{\partial\phi}\right)-2\cos\phi\frac{\partial^2\psi_0}{\partial\rho\,\partial\phi}-2\rho\cos\phi\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\psi_0}{\partial\phi}\right)+2\sin\phi\frac{\partial\psi_0}{\partial\rho}\right\}\right],$$

$$P^{(2)}_{\rho\rho} = \left[-p_2+2E\left\{\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\psi_2}{\partial\phi}\right)-2\cos\phi\frac{\partial^2\psi_1}{\partial\rho\,\partial\phi}+\frac{\partial}{\partial\rho}\left(\frac{\rho^2+1}{\rho}\frac{\partial\psi_0}{\partial\phi}\right)\right.\right.$$

$$\left.\left.-2\rho\cos\phi\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\psi_1}{\partial\phi}\right)+4\rho\cos^2\phi\frac{\partial^2\psi_0}{\partial\rho\,\partial\phi}+(\rho^2+1)\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\psi_0}{\partial\phi}\right)+2\sin\phi\frac{\partial\psi_1}{\partial\rho}\right\}\right];$$

$$P_{\phi\rho} = P^{(0)}_{\phi\rho}+\delta P^{(1)}_{\phi\rho}+\delta^2 P^{(2)}_{\phi\rho}+\dots,$$

where

$$P^{(0)}_{\phi\rho} = E\left[-\rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\psi_0}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\psi_0}{\partial\phi^2}\right],$$

$$P^{(1)}_{\phi\rho} = E\left[-\rho\frac{\partial}{\partial\rho}\frac{1}{\rho}\left(\frac{\partial\psi_1}{\partial\rho}-4\rho\cos\phi\frac{\partial\psi_0}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial}{\partial\phi}\left(\frac{\partial\psi_1}{\partial\phi}-4\rho\cos\phi\frac{\partial\psi_0}{\partial\phi}\right)\right],$$

$$P^{(2)}_{\phi\rho} = E\left[-\rho\frac{\partial}{\partial\rho}\frac{1}{\rho}\left(\frac{\partial\psi_2}{\partial\rho}-4\rho\cos\phi\frac{\partial\psi_1}{\partial\rho}+2(2\rho^2\cos^2\phi+\rho^2+1)\frac{\partial\psi_0}{\partial\rho}\right)\right.$$

$$\left.+\frac{1}{\rho^2}\frac{\partial}{\partial\phi}\left(\frac{\partial\psi_2}{\partial\phi}-4\rho\cos\phi\frac{\partial\psi_1}{\partial\phi}+2(2\rho^2\cos^2\phi+\rho^2+1)\frac{\partial\psi_0}{\partial\phi}\right)\right].$$

Appendix B

The perturbation analysis described in § 3 is developed here in the minimal detail for it to be followed and reconstructed.

The substitution of the series expansions (20) into the basic equations (12) and (13), the boundary conditions (14) and (15) and the constraints (18) and (19) leads to a sequence of boundary-value problems associated with consecutive powers of δ. The fundamental problem, that for $O(\delta^0)$ terms, is

$$\frac{1}{\rho}\left[\left(\frac{\partial}{\partial\rho}\Delta\psi_0\right)\frac{\partial\psi_0}{\partial\phi}-\frac{\partial}{\partial\phi}(\Delta\psi_0)\frac{\partial\psi_0}{\partial\rho}\right] = E\Delta\Delta\psi_0, \tag{B 1}$$

with the conditions

$$\left.\begin{aligned}\psi_0 &= 0, \quad \partial\psi_0/\partial\rho = -\gamma_0 \quad \text{at} \quad \rho = 1;\\ \psi_0 &= Q_0, \quad \partial\psi_0/\partial\rho = -\beta \quad \text{at} \quad \rho = \beta.\end{aligned}\right\} \tag{B 2}$$

The pressure is determined from

$$-\Delta\psi_0\nabla\psi_0 = -\nabla R_0 + E\nabla\Delta\psi_0\times\hat{\mathbf{k}}, \tag{B 3}$$

where
$$R_0 = p_0+\tfrac{1}{2}\nabla\psi_0\,.\,\nabla\psi_0. \tag{B 4}$$

The conditions of zero net force and zero torque on the straw, $\rho = 1$, imply that

$$\int_0^{2\pi}(P^{(0)}_{\rho\rho}+iP^{(0)}_{\phi\rho})e^{i\phi}d\phi = 0 \tag{B 5}$$

and
$$\int_0^{2\pi}P^{(0)}_{\rho\phi}d\phi = 0. \tag{B 6}$$

The solution of this problem is

$$\left.\begin{aligned}\psi_0 &= -\tfrac{1}{2}(\rho^2-1),\\ p_0 &= -\tfrac{1}{2}\rho^2+\text{constant},\end{aligned}\right\} \tag{B 7}$$

with
$$\gamma_0 = 1, \quad Q_0 = -\tfrac{1}{2}(\beta^2-1);$$

and the motion to this degree of approximation is essentially a state of rigid rotation. No other unknown quantities can be determined at this stage and the theory is carried to the next order.

The new boundary-value problem, $O(\delta)$, is

$$\begin{aligned}\frac{1}{\rho}\Bigg[\frac{\partial}{\partial\rho}(\Delta\psi_0)\frac{\partial\psi_1}{\partial\phi}-\frac{\partial}{\partial\phi}(\Delta\psi_0)\frac{\partial\psi_1}{\partial\rho}+\frac{\partial}{\partial\rho}(\Delta\psi_1-4\rho\cos\phi\Delta\psi_0)\frac{\partial\psi_0}{\partial\phi}\\ -\frac{\partial}{\partial\phi}(\Delta\psi_1-4\rho\cos\phi\Delta\psi_0)\frac{\partial\psi_0}{\partial\rho}\Bigg] = E\{\Delta\Delta\psi_1-4\Delta(\rho\cos\phi\Delta\psi_0)\},\end{aligned} \tag{B 8}$$

with
$$\left.\begin{aligned}\psi_1 &= 0, \quad \partial\psi_1/\partial\rho = -(\gamma_1+2\gamma_0\cos\phi) \quad \text{at} \quad \rho = 1;\\ \psi_1 &= Q_1, \quad \partial\psi_1/\partial\rho = -2\beta^2\cos\phi \quad \text{at} \quad \rho = \beta;\end{aligned}\right\} \tag{B 9}$$

at $\rho = 1$
$$\left.\begin{aligned}\int_0^{2\pi}(2\mathscr{F}_0e^{2i\phi}+\mathscr{F}_1e^{i\phi})d\phi &= 0,\\ \int_0^{2\pi}(2\cos\phi P^{(0)}_{\rho\phi}+P^{(1)}_{\rho\phi})d\phi &= 0.\end{aligned}\right\} \tag{B 10}$$

Here we have for compactness defined

$$\mathscr{F}_n = P^{(n)}_{\rho\rho} + iP^{(n)}_{\phi\rho}.$$

The equation for the pressure is

$$-[\Delta\psi_0 \nabla\psi_1 + (\Delta\psi_1 - 4\rho\cos\phi\Delta\psi_0)\nabla\psi_0] = -\nabla R_1 + E\nabla(\Delta\psi_1 - 4\rho\cos\phi\Delta\psi_0)\times\hat{\mathbf{k}}, \quad \text{(B 11)}$$

where

$$R_1 = p_1 + \alpha_1 \sin(\omega_0 + \phi) + \tfrac{1}{2}(2\nabla\psi_0 . \nabla\psi_1 - 4\rho\cos\phi\nabla\psi_0 . \nabla\psi_0). \quad \text{(B 12)}$$

These equations are considerably simplified by substituting the explicit formulae for ψ_0, in particular $\partial\psi_0/\partial\phi = 0$ and $\Delta\psi_0 = -2$, and by introducing the new variable

$$\Psi_1 = \psi_1 + \rho^3\cos\phi. \quad \text{(B 13)}$$

Equations (B 8)–(B 10) then reduce to

$$\partial(\Delta\Psi_1)/\partial\phi = E\Delta\Delta\Psi_1; \quad \text{(B 14)}$$

on $\rho = 1$: $\quad \Psi_1 = \cos\phi, \quad \partial\Psi_1/\partial\rho = -\gamma_1 + \cos\phi,$

$$\int_0^{2\pi} \frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\Psi_1}{\partial\rho}\right) d\phi = 0,$$

$$\int_0^{2\pi} \left[-p_1 + iE\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\Psi_1\right)\right] e^{i\phi}\, d\phi = 0;$$

on $\rho = \beta$: $\quad \Psi_1 = Q_1 + \beta^3\cos\phi, \quad \partial\Psi_1/\partial\rho = \beta^2\cos\phi.$

The pressure is obtained from

$$(\Delta\Psi_1)\rho\hat{\boldsymbol{\rho}} = -\nabla(R_1 + 2\psi_1) + E\nabla(\Delta\Psi_1)\times\hat{\mathbf{k}}. \quad \text{(B 15)}$$

Boundary-layer methods are used to solve this problem; conditions on the normal derivatives of the stream function require that the solution be determined correct to $O(E^{\frac{1}{2}})$. If in the interior the stream function is developed as

$$\Psi_1 = \Psi_{10} + E^{\frac{1}{2}}\Psi_{11} + E\Psi_{12} + \dots$$

then from the basic equation we find that

$$\Delta\Psi_{1n} = f_n(\rho)$$

for $n = 0, 1, 2$ with $\Delta f_n(\rho) = 0$. More simply put

$$\Delta\Psi_1 = f(\rho).$$

Since pressure and velocity must be periodic in ϕ, the integration of the ϕ component of the momentum equation (B 15) yields

$$\int_0^{2\pi} \frac{\partial}{\partial\rho}(\Delta\Psi_1)\, d\phi = 0,$$

so that in fact

$$\Delta\Psi_1 = \text{constant}, \quad \text{(B 16)}$$

outside boundary layers. Conventional techniques are now applied to obtain the following approximations:

$$\psi_1 \simeq \mathrm{Re}\,[(\{(1+\beta^2)\rho - \beta^2\rho^{-1} - \rho^3\} + E^{\frac{1}{2}}\{G_1(\rho) + \tilde{G}_i(\xi) + \tilde{G}_f(\eta)\})e^{i\phi}], \quad \text{(B 17)}$$

where
$$\xi = E^{-\frac{1}{2}}(\rho-1), \quad \eta = E^{-\frac{1}{2}}(\beta-\rho),$$
$$G_1(\rho) = \frac{2\beta}{\beta-1} e^{-\frac{1}{4}i\pi}[\rho-\rho^{-1}(\beta^2-\beta+1)],$$
$$\tilde{G}_i(\xi) = 2\beta^2 e^{-\frac{1}{4}i\pi}\exp[-i^{\frac{1}{2}}\xi],$$
$$\tilde{G}_f(\eta) = -2e^{-\frac{1}{4}i\pi}\exp[-i^{\frac{1}{2}}\eta];$$
$$p_1 \simeq [-(1+\beta^2)\rho+3\beta^2\rho^{-1}+\rho^3-2\beta^2]\cos\phi. \tag{B 18}$$
In addition
$$\alpha_1 = -2\beta^2, \quad \omega_0 = -\tfrac{1}{2}\pi, \quad \gamma_1 = 0, \quad Q_1 = 0. \tag{B 19}$$

The equations of the second-order boundary-value problem are
$$\begin{aligned}\frac{1}{\rho}\bigg[&\frac{\partial}{\partial\rho}(\Delta\psi_0)\frac{\partial\psi_2}{\partial\phi}-\frac{\partial}{\partial\phi}(\Delta\psi_0)\frac{\partial\psi_2}{\partial\rho}+\frac{\partial}{\partial\rho}(\Delta\psi_1-4\rho\cos\phi\Delta\psi_0)\frac{\partial\psi_1}{\partial\phi}\\&-\frac{\partial}{\partial\phi}(\Delta\psi_1-4\rho\cos\phi\Delta\psi_0)\frac{\partial\psi_1}{\partial\rho}+\frac{\partial}{\partial\rho}\{\Delta\psi_2-4\rho\cos\phi\Delta\psi_1\\&+2\Delta\psi_0(2\rho^2\cos^2\phi+\rho^2+1)\}\frac{\partial\psi_0}{\partial\phi}\\&-\frac{\partial}{\partial\phi}\{\Delta\psi_2-4\rho\cos\phi\Delta\psi_1+2\Delta\psi_0(2\rho^2\cos^2\phi+\rho^2+1)\}\frac{\partial\psi_0}{\partial\rho}\bigg]\\&=E\Delta[\Delta\psi_2-4\rho\cos\phi\Delta\psi_1+2(2\rho^2\cos^2\phi+\rho^2+1)\Delta\psi_0];\end{aligned} \tag{B 20}$$
$$\begin{aligned}-[\Delta\psi_0\nabla\psi_2+(\Delta\psi_1-4\rho\cos\phi\Delta\psi_0)\nabla\psi_1+\{\Delta\psi_2-4\rho\cos\phi\Delta\psi_1&\\+2\Delta\psi_0(2\rho^2\cos^2\phi+\rho^2+1)\}\nabla\psi_0]&\\=-\nabla R_2+E\nabla[\Delta\psi_2-4\rho\cos\phi\Delta\psi_1+2\Delta\psi_0(2\rho^2\cos^2\phi+\rho^2+1)]\times\hat{\mathbf{k}},&\end{aligned} \tag{B 21}$$
where
$$\begin{aligned}R_2 = p_2&+\alpha_1[2\rho\cos\phi\sin(\omega_0+\phi)+\omega_1\cos(\omega_0+\phi)-(\rho^2+\beta^2)\sin\omega_0]+\alpha_2\sin(\omega_0+\phi)\\&+\tfrac{1}{2}[\nabla\psi_1.\nabla\psi_1+2\nabla\psi_0.\nabla\psi_2-8\rho\cos\phi\nabla\psi_0.\nabla\psi_1\\&+2(2\rho^2\cos^2\phi+\rho^2+1)\nabla\psi_0.\nabla\psi_0].\end{aligned} \tag{B 22}$$

The boundary conditions are as follows:

on $\rho = 1$:
$$\left.\begin{aligned}&\psi_2 = 0, \quad \partial\psi_2/\partial\rho = -(\gamma_2+2\gamma_1\cos\phi+2\gamma_0\cos 2\phi),\\&\int_0^{2\pi}(P^{(0)}_{\rho\phi}(4\cos^2\phi-2)+2\cos\phi P^{(1)}_{\rho\phi}+P^{(2)}_{\rho\phi})\,d\phi = 0,\\&\int_0^{2\pi}(\mathscr{F}_2e^{i\phi}+2\mathscr{F}_1e^{2i\phi}+2\mathscr{F}_0(3e^{3i\phi}-e^{i\phi}))\,d\phi = 0;\end{aligned}\right\} \tag{B 23}$$
on $\rho = \beta$:
$$\psi_2 = Q_2, \quad \partial\psi_2/\partial\rho = -2\beta(\beta^2\cos 2\phi+\beta^2-1). \tag{B 24}$$

This problem also simplifies by using information already in hand and by introducing
$$\Psi_2 = \psi_2+\rho^4\cos 2\phi+\tfrac{1}{2}\rho^4-\rho^2. \tag{B 25}$$

We obtain the equations
$$\begin{aligned}\frac{1}{\rho}\left[\left(\frac{\partial}{\partial\rho}\Delta\Psi_1\right)\left(\frac{\partial\Psi_1}{\partial\phi}+\rho^3\sin\phi\right)-\left(\frac{\partial\Delta\Psi_1}{\partial\phi}\right)\left(\frac{\partial\Psi_1}{\partial\rho}-3\rho^2\cos\phi\right)\right]&\\+\frac{\partial}{\partial\phi}(\Delta\Psi_2-4\rho\cos\phi\Delta\Psi_1) = E\Delta(\Delta\Psi_2-4\rho\cos\phi\Delta\Psi_1)&\end{aligned} \tag{B 26}$$

and

$$\rho(\Delta\Psi_2 - 4\rho\cos\phi\Delta\Psi_1)\,\hat{\boldsymbol{\rho}} = \{-\nabla(R_2 + 2\psi_2) + \Delta\Psi_1\nabla\psi_1\} + E\nabla(\Delta\Psi_2 - 4\rho\cos\phi\Delta\Psi_1)\times\hat{\mathbf{k}}. \quad \text{(B 27)}$$

Once again boundary-layer theory is employed to solve the problem. Since by (B 16) $\Delta\Psi_1$ is constant to $O(E^{\frac{3}{4}})$ in the interior domain, there the braced quantity in (B 27) is representable entirely as a gradient, while (B 26) in the interior reduces to

$$\partial(\Delta\Psi_2)/\partial\phi = E\Delta\Delta\Psi_2.$$

The arguments that lead to (B 16) can be repeated to show that in the 'inviscid' interior $\Delta\Psi_2 =$ constant, or in particular

$$\Psi_2 = C_0\rho^2 + C_1 + C_2\ln\rho + \sum_1^{\infty}\left(A_n\rho^n + \frac{B_n}{\rho^n}\right)e^{in\theta}. \quad \text{(B 28)}$$

From the boundary conditions

$$\Psi_2 = \begin{cases}\cos 2\phi - \frac{1}{2} & \text{at} \quad \rho = 1,\\ Q_2 + \beta^4\cos 2\phi + \frac{1}{2}\beta^4 - \beta^2 & \text{at} \quad \rho = \beta,\end{cases}$$

we find that

$$A_n = B_n = 0, \quad n \neq 2;$$

$$A_2 = \frac{\beta^4 + \beta^2 + 1}{\beta^2 + 1}, \quad B_2 = \frac{-\beta^4}{\beta^2 + 1}, \quad \text{(B 29)}$$

and

$$C_0 + C_1 = -\tfrac{1}{2},$$

$$C_0\beta^2 + C_1 + C_2\ln\beta = Q_2 + \tfrac{1}{2}\beta^4 - \beta^2.$$

If the functions Ψ_1, Ψ_2 are resolved into interior and boundary-layer components as

$$\Psi_1 = \bar{\Psi}_1 + E^{\frac{1}{2}}\tilde{\Psi}_1(\xi,\phi), \quad \Psi_2 = \bar{\Psi}_2 + E^{\frac{1}{2}}\tilde{\Psi}_2(\xi,\phi), \quad \text{(B 30)}$$

$$\xi = E^{-\frac{1}{2}}(\rho - 1), \quad \text{(B 31)}$$

the appropriate form of (B 26) in the boundary layer is then

$$\frac{\partial^3\tilde{\Psi}_1}{\partial\xi^3}[2(1-\beta^2)\xi\sin\phi + \operatorname{Re} i(G_1(1) + \tilde{G}_i(\xi))e^{i\phi}] - \frac{\partial^3\tilde{\Psi}_1}{\partial\xi^2\partial\phi}\left[2(\beta^2-1)\cos\phi + \frac{\partial\tilde{\Psi}_1}{\partial\xi}\right] + \frac{\partial^3}{\partial\xi^2\partial\phi}[\tilde{\Psi}_2 - 4\cos\phi\tilde{\Psi}_1] = \frac{\partial^4}{\partial\xi^4}(\tilde{\Psi}_2 - 4\cos\phi\,\tilde{\Psi}_1). \quad \text{(B 32)}$$

The boundary conditions at $\rho = 1$ on tangential velocity and torque, (B 23), imply that

$$\frac{\partial}{\partial\rho}\bar{\Psi}_2(1,\phi) + \frac{\partial}{\partial\xi}\tilde{\Psi}_2(0,\phi) = -(\gamma_2 + 2\cos 2\phi) \quad \text{(B 33)}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\left(2\cos\phi\frac{\partial^2\tilde{\Psi}_1}{\partial\xi^2} - \frac{\partial^2\tilde{\Psi}_2}{\partial\xi^2}\right)d\phi = E^{\frac{1}{2}}\gamma_2. \quad \text{(B 34)}$$

The remaining condition at $\rho = 1$ and the conditions and the boundary-layer equations at $\rho = \beta$ must still be specified. However solution of the complete problem is a formidable task and since there is available sufficient information to pluck from the theory the quantities sought, γ_2 and $\overline{\Psi}_\rho(1,\phi)$, we make this limited calculation the sole objective here. The basic technique to this end is to consider zonal averages, as for example,

$$\langle\Psi_2\rangle = \frac{1}{2\pi}\int_0^{2\pi}\Psi_2\,d\phi. \tag{B 35}$$

With the explicit form for $\tilde{\Psi}_1$ obtainable from (B 17), the zonal average and integration of (B 32) yield after much algebra

$$\frac{\partial^2}{\partial\xi^2}\langle\tilde{\Psi}_2\rangle = \mathrm{Re}\{-\tfrac{1}{2}(-i)^{\frac{1}{2}}[-2\beta^2 e^{\frac{1}{4}i\pi}G_i^*(\xi)+\tfrac{1}{2}|G_i|^2+2i(\beta^2-1)(\xi+2(-i)^{-\frac{1}{2}})G_i^*] \\ +i(\beta^2+3)G_i^*-\tfrac{1}{4}i^{\frac{1}{2}}|G_i|^2\}, \tag{B 36}$$

where an asterisk denotes the complex conjugate. Upon evaluating this expression at $\zeta = 0$ and substituting the result in (B 34) it follows that

$$\gamma_2 = -2^{\frac{1}{2}}E^{-\frac{1}{2}}\beta^2(\beta^2-4), \tag{B 37}$$

or for β^2 large

$$\gamma_2 \simeq -2^{\frac{1}{2}}E^{-\frac{1}{2}}\beta^4. \tag{B 38}$$

The integration of (B 36) yields, in particular,

$$\frac{\partial}{\partial\xi}\langle\tilde{\Psi}_2\rangle|_0 = 3\beta^2(\beta^2-4).$$

This formula is substituted into the average of (B 33)

$$\left\langle\frac{\partial}{\partial\rho}\overline{\Psi}_2(1,\phi)\right\rangle+\frac{\partial}{\partial\xi}\langle\tilde{\Psi}_2(0,\phi)\rangle = -\gamma_2$$

to obtain

$$\partial\langle\overline{\Psi}_2(1,\phi)\rangle/\partial\rho = -\gamma_2-3\beta^2(\beta^2-4).$$

But from (B 28)

$$\frac{\partial}{\partial\rho}\overline{\Psi}_2(1,\phi) = \frac{\partial}{\partial\rho}\langle\overline{\Psi}_2(1,\phi)\rangle+2\left(\frac{2\beta^4+\beta^2+1}{\beta^2+1}\right)\cos 2\phi.$$

From the last two equations we conclude that

$$\begin{aligned}\frac{\partial}{\partial\rho}\overline{\psi}_2(1,\phi) &= \frac{\partial}{\partial\rho}\overline{\Psi}_2(1,\phi)-4\cos 2\phi \\ &= -\gamma_2-3\beta^2(\beta^2-4)+2\frac{(2\beta^2+1)(\beta^2-1)}{\beta^2+1}\cos 2\phi.\end{aligned}$$

Therefore for β large and E small, see equation (B38),

$$\partial\overline{\psi}_2(1,\phi)/\partial\rho \simeq -\gamma_2.$$

And finally we may combine our results to obtain the approximation

$$\partial\overline{\psi}(1,\phi)/\partial\rho \simeq -1+2\beta^2\delta\cos\phi-\gamma_2\delta^2.$$

This completes the derivation of the formulae quoted in the text.

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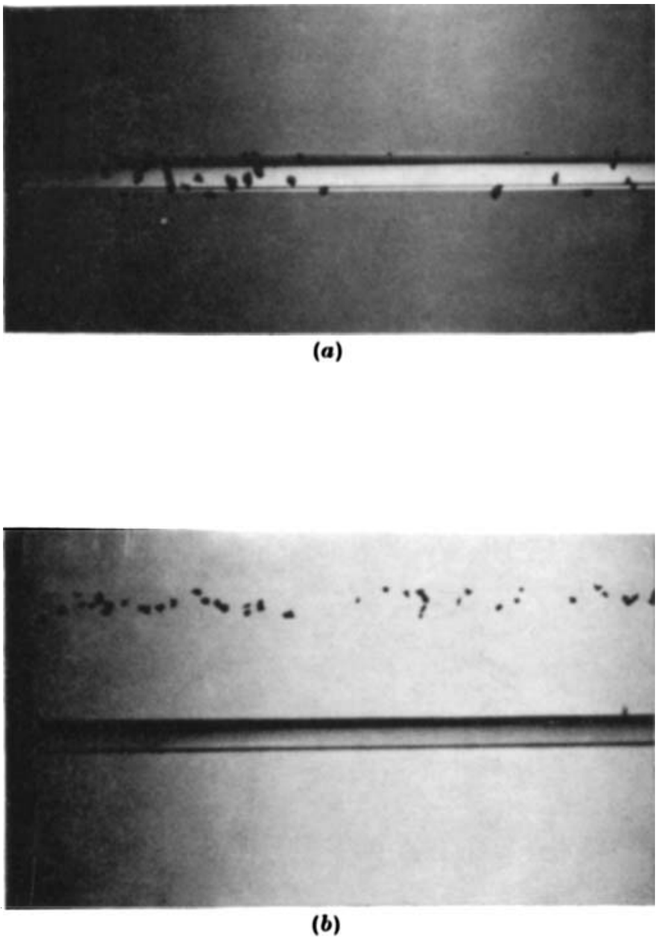

(a)

(b)

FIGURE 1. (*a*) Floats on the surface of the straw in stable conditions. (*b*) Instability; floats are aligned along the axis of a detached vertex.